

# A PROBABILISTIC APPROACH TO SYSTEMS OF PARAMETERS AND NOETHER NORMALIZATION

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**ABSTRACT.** We study systems of parameters over finite fields from a probabilistic perspective. Our central technique is an adaptation of Poonen's closed point sieve, where we sieve over higher dimensional subvarieties, and we express the desired probabilities via a zeta function-like power series that enumerates higher dimensional varieties instead of closed points. Applications include an effective Noether normalization result over a finite field, and a new proof of a recent result of Gabber-Liu-Lorenzini and Chinburg-Moret-Bailly-Pappas-Taylor on uniform Noether normalizations for projective families over the integers.

Given an  $n$ -dimensional projective scheme  $X \subseteq \mathbb{P}^r$  over a field, Noether normalization says that we can find homogeneous polynomials that induce a finite morphism  $X \rightarrow \mathbb{P}^n$ . Such a morphism is determined by a system of parameters, namely by choosing homogeneous polynomials  $f_0, f_1, \dots, f_n$  of degree  $d$  where  $X \cap V(f_0, f_1, \dots, f_n) = \emptyset$ . Over an infinite field any generic choice of polynomials will work. Over a finite field, we ask: *what is the probability that a random choice of polynomials will yield such a finite morphism?*

Answering this question involves studying the distribution of systems of parameters from both a geometric and probabilistic viewpoint. It is also useful to analyze partial systems of parameters, so for  $k \leq n$  we say that  $f_0, f_1, \dots, f_k$  are **parameters** on  $X$  if

$$\dim \mathbb{V}(f_0, f_1, \dots, f_k) \cap X = \dim X - (k + 1).$$

By convention, the empty set has dimension  $-1$ .

For the geometric side, we fix a field  $\mathbf{k}$  and let  $S = \mathbf{k}[x_0, \dots, x_r]$  be the coordinate ring of  $\mathbb{P}_{\mathbf{k}}^r$ . We write  $S_d$  for the vector space of degree  $d$  polynomials in  $S$ . In §4, we define a scheme  $\mathcal{D}_{k,d}(X)$  parametrizing collections that do not form parameters. The  $\mathbf{k}$ -points of  $\mathcal{D}_{k,d}(X)$  are

$$\mathcal{D}_{k,d}(X)(\mathbf{k}) = \{(f_0, f_1, \dots, f_k) \text{ that are **not** parameters on } X\} \subset \underbrace{S_d \times \dots \times S_d}_{k+1 \text{ copies}}.$$

We bound the codimension of these closed subschemes of the affine space  $S_d^{\oplus k+1}$ .

**Theorem A.** *Let  $X \subseteq \mathbb{P}_{\mathbf{k}}^r$  be an  $n$ -dimensional closed subscheme. We have:*

$$\text{codim } \mathcal{D}_{k,d}(X) = \begin{cases} \geq \binom{n-k+d}{n-k} & \text{if } k < n \\ = 1 & \text{if } k = n. \end{cases}$$

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This generalizes several results from the literature: the case  $k = n$  is a classical result about Chow forms [GKZ08, 3.2.B]; the case  $d = 1$  is a classical result about determinantal varieties [Mac94]; and the case  $k = 0$  appears in [Ben11, Lemme 2.3]. If  $k < n$ , then the codimension grows as  $d \rightarrow \infty$  and this factors into our asymptotic analysis over finite fields. It also leads to a uniform convergence result that allows us to go from a finite field to  $\mathbb{Z}$ .

For the probabilistic side, we work over a finite field  $\mathbb{F}_q$  and compute the asymptotic probability that random polynomials  $(f_0, f_1, \dots, f_k)$  of degree  $d$  are parameters on  $X$ . This is inspired by [Poo04], and it forms the heart of the paper. There is a bifurcation between the maximal and submaximal cases, reflecting Theorem A.

**Theorem B.** *Let  $X \subseteq \mathbb{P}_{\mathbb{F}_q}^r$  be an  $n$ -dimensional closed subscheme. Then the asymptotic probability that random polynomials  $(f_0, f_1, \dots, f_k)$  of degree  $d$  are parameters on  $X$  is*

$$\lim_{d \rightarrow \infty} \text{Prob} \left( \begin{array}{l} (f_0, \dots, f_k) \text{ of degree } d \\ \text{are parameters on } X \end{array} \right) = \begin{cases} 1 & \text{if } k < n \\ \zeta_X(n+1)^{-1} & \text{if } k = n \end{cases}$$

where  $\zeta_X(s)$  is the arithmetic zeta function of  $X$ .

The maximal case  $k = n$  is due to Bucur and Kedlaya [BK12, Theorem 1.2], and is proven using Poonen's closed point sieve. For submaximal cases where  $k < n$ , we adapt Poonen's technique by sieving over closed subvarieties of dimension  $n - k$ . Just as the case  $k = n$  depends on the zeta function of  $X$ , which counts points in  $X$  of varying degrees, we show that each case  $k < n$  depends on a power series that counts  $(n - k)$ -dimensional subvarieties of varying degrees. The full computation of these probabilities appears in Theorem 6.2, while the following corollary of Theorem 6.2 computes the first error term.

**Corollary 1.1.** *Let  $X \subseteq \mathbb{P}_{\mathbb{F}_q}^r$  be a  $n$ -dimensional closed subscheme and let  $k < n$ . Then*

$$\lim_{d \rightarrow \infty} \frac{\text{Prob} \left( \begin{array}{l} (f_0, \dots, f_k) \text{ of degree } d \\ \text{are not parameters on } X \end{array} \right)}{q^{-(k+1)\binom{n-k+d}{n-k}}} = \# \left\{ \begin{array}{l} (n-k)\text{-planes } L \subseteq \mathbb{P}_{\mathbb{F}_q}^r \\ \text{such that } L \subseteq X \end{array} \right\}.$$

When  $k < n$ , the probability in Theorem B behaves like  $1 - O\left(q^{-(k+1)\binom{n-k+d}{n-k}}\right)$ , with the precise convergence governed by the number of linear spaces of dimension  $n - k$  in  $X$ . It is thus more difficult to find parameters when  $X$  contains lots of linear spaces, see Example 8.1.

The main difficulty in Theorem B and Theorem 6.2 is bounding the error of the sieve. We control this error via a uniform lower bound for Hilbert functions provided in Lemma 3.1.

Our probabilistic results enable us to give an effective bound for Noether normalization over a finite field. The bound is in terms of the sum of the degrees of the irreducible components. If  $X \subseteq \mathbb{P}^r$  has minimal irreducible components  $V_1, \dots, V_s$  (considered with the reduced scheme structure), then we define  $\widehat{\deg}(X) := \sum_{i=1}^s \deg(V_i)$  (see also Definition 2.2.)

**Corollary 1.2.** *Let  $X \subseteq \mathbb{P}_{\mathbb{F}_q}^r$  where  $\dim X = n$ . If  $\max\{d, \frac{q}{d^{n+1}}\} \geq \widehat{\deg}(X)$  and*

$$d > \log_q \widehat{\deg}(X) + \log_q n + n \log_q d$$

*then there exist  $f_0, f_1, \dots, f_n$  of degree  $\max\{d, d^n\}$  inducing a finite morphism  $\pi : X \rightarrow \mathbb{P}_{\mathbb{F}_q}^n$ .*

The bound is asymptotically optimal in  $q$ . Namely, if we fix  $\widehat{\deg}(X)$ , then as  $q \rightarrow \infty$ , the bound becomes  $d = 1$ . Thus, linear Noether normalizations exist if  $q \gg \widehat{\deg}(X)$ . For a fixed  $q$ , we expect the bound could be significantly improved. This is interesting even in the case  $\dim X = 0$ , where it is related to Kakeya type problems over finite fields [EE16, EOT10].

Corollary 1.2 provides the first explicit bound for Noetherian normalization over a finite field. (One could potentially derive an explicit bound from Nagata's argument in [Nag62, Chapter I.14], though the inductive nature of that construction would at best yield a bound that is multiply exponential in the largest degree of a defining equation of  $X$ .)

After computing the probabilities over finite fields, we combine these analyses and characterize the distribution of parameters on projective  $B$ -schemes where  $B = \mathbb{Z}$  or  $\mathbb{F}_q[t]$ . We use standard notions of density for a subset of a free  $B$ -module; see Definition 7.1.

**Theorem C.** *If  $X \subseteq \mathbb{P}_B^r$  is a closed subscheme whose general fiber over  $B$  has dimension  $n$ , then*

$$\lim_{d \rightarrow \infty} \text{Density} \left\{ \begin{array}{l} (f_0, f_1, \dots, f_k) \text{ of degree } d \text{ that restrict} \\ \text{to parameters on } X_p \text{ for all } p \end{array} \right\} = \begin{cases} 1 & \text{if } k < n \\ 0 & \text{if } k = n \text{ and all } d. \end{cases}$$

The density over  $B$  thus equals the product over all the fibers of the asymptotic probabilities over  $\mathbb{F}_q$ . In the case  $B = \mathbb{Z}$ , our proof relies on Ekedahl's infinite Chinese Remainder Theorem [Eke91, Theorem 1.2] combined with Proposition 5.1, which illustrates uniform convergence in  $p$  for the asymptotic probabilities in Theorem B. In the case  $B = \mathbb{F}_q[t]$ , we use Poonen's analogue of Ekedahl's result [Poo03, Theorem 3.1].

When  $k = n$ , an analogue of Theorem C for smoothness is given by Poonen's [Poo04, Theorem 5.13]. Moreover, it is believed that there are no smooth hypersurfaces of degree  $> 2$  over  $\mathbb{Z}$ . By contrast, the density zero subset from Theorem C turns out to always be nonempty. This leads to a new proof of a recent result about uniform Noether normalizations.

**Corollary 1.3.** *Let  $B = \mathbb{Z}$  or  $\mathbb{F}_q[t]$ . Let  $X \subseteq \mathbb{P}_B^r$  be a closed subscheme. If each fiber of  $X$  over  $B$  has dimension  $n$ , then for some  $d$ , there exist homogeneous polynomials  $f_0, f_1, \dots, f_n \in B[x_0, x_1, \dots, x_r]$  of degree  $d$  inducing a finite morphism  $\pi : X \rightarrow \mathbb{P}_B^n$ .*

Theorem C shows that the collections defining a finite map  $\pi$  have density zero, even as  $d \rightarrow \infty$ . Thus the existence of  $\pi$  is subtle and perhaps unexpected. Corollary 1.3 is a special case of a recent result of Chinburg-Moret-Bailly-Pappas-Taylor [CMBPT12, Theorem 1.2] and of Gabber-Liu-Lorenzini [GLL15, Theorem 8.1]. Our proof of the corollary involves two steps. We first use Theorem C to choose a submaximal collection  $f_0, f_1, \dots, f_{n-1}$  of parameters on  $X$ . This yields a scheme  $X' := X \cap \mathbb{V}(f_0, f_1, \dots, f_{n-1})$  with 0-dimensional fibers over  $\text{Spec}(B)$ , and we then use that  $\text{Pic}(X')$  is torsion to find the final section  $f_n$ .

Corollary 1.3 does not require the full strength of Theorem C, as we only need that the set of good choices for  $f_0, f_1, \dots, f_{n-1}$  is nonempty, and the proofs in [CMBPT12, GLL15] do use simpler techniques when selecting  $f_0, f_1, \dots, f_{n-1}$ . For the final section  $f_n$ , they use essentially the same technique as we do. Since we rely on the infinite Chinese Remainder Theorems of Ekedahl and Poonen, Corollary 1.3 only recovers the results in [CMBPT12, GLL15] for  $\mathbb{Z}$  and  $\mathbb{F}_q[t]$ , but not for more general number fields or function fields.

Corollary 1.3 can fail when  $B$  is any of  $\mathbb{Q}[t]$  or  $\mathbb{Z}[t]$  or  $\mathbb{F}_q[s, t]$ , as in those cases, the Picard group of a finite cover of  $\operatorname{Spec} B$  can fail to be torsion. See §8 for explicit examples and counterexamples and see [CMBPT12, GLL15] for generalizations and applications.

It would be interesting to produce an effective version of Corollary 1.3 similar to Corollary 1.2. One would need to bound the height of  $f_0, f_1, \dots, f_{n-1}$  when applying Theorem C, and combining this with effective bounds on the size of the class group of a number field.

There are a few earlier results related to Noether normalization over the integers. For instance [Moh79] shows that Noether normalizations of semigroup rings always exist over  $\mathbb{Z}$ ; and [Nag62, Theorem 14.4] implies that given a family over any base, one can find a Noether normalization over an open subset of the base. Relative Noether normalizations play a key role in [Ach15, §5]. There is also the incorrect claim in [ZS75, p. 124] that Noether normalizations exist over any infinite base ring (see [AK07]). Brennan and Epstein [BE11] analyze the distribution of systems of parameters from a different perspective, introducing the notion of a generic matroid to relate various different systems of parameters.

This paper is organized as follows. §2 gathers background results and §3 involves a key lower bound on Hilbert functions. §4 contains our geometric analysis of parameters including a proof of Theorem A. §5 and §6 contain our probabilistic analysis of parameters over finite fields: §5 proves Theorem B while §6 gives the more detailed description via an analogue of the zeta function enumerating  $(n - k)$ -dimensional subvarieties. §7 contains our analysis over  $\mathbb{Z}$  including proofs of Theorems C and 1.3 and related corollaries. §8 contains examples.

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## 2. BACKGROUND

In this section, we gather some algebraic and geometric facts that we will cite throughout.

**Lemma 2.1.** *Let  $\mathbf{k}$  be a field and let  $R$  be a  $(k + 1)$ -dimensional graded  $\mathbf{k}$ -algebra where  $R_0 = \mathbf{k}$ . If  $f_0, f_1, \dots, f_k$  are homogeneous elements of degree  $d$  and  $R/(f_0, f_1, \dots, f_k)$  has finite length, then the extension  $\mathbf{k}[z_0, z_1, \dots, z_k] \rightarrow R$  given by  $z_i \mapsto f_i$  is a finite extension.*

*Proof.* See [BH93, Theorem 1.5.17]. □

**Definition 2.2.** *Let  $X \subseteq \mathbb{P}^r$  be a projective scheme with minimal irreducible components  $V_1, \dots, V_s$  (considered with the reduced scheme structure). We define  $\widehat{\deg}(X) := \sum_{i=1}^s \deg(V_i)$ . For a subscheme  $X' \subseteq \mathbb{A}^r$  with projective closure  $\overline{X'} \subseteq \mathbb{P}^r$  we define  $\widehat{\deg}(X') := \widehat{\deg}(\overline{X'})$ .*

This provides a notion of degree which ignores nonreduced structure but takes into account components of lower dimension. Similar definitions have appeared in the literature: for

instance, in the language of [BM93, §3], we would have  $\widehat{\deg}(X) = \sum_{j=0}^{\dim X} \text{geom-deg}_j(X)$ . This definition is useful when bounding the number of points of a scheme over a finite field.

**Lemma 2.3.** *Let  $X \subseteq \mathbb{P}_{\mathbb{F}_q}^r$  be a closed subscheme, where  $\mathbb{F}_q$  is a finite field. Then*

$$\#X(\mathbb{F}_q) \leq \widehat{\deg}(X)q^{\dim X}.$$

*Proof.* This is an immediate consequence of the Schwarz-Zippel lemma, which implies that for an irreducible algebraic variety  $V_i \subseteq \mathbb{P}_{\mathbb{F}_q}^r$  we have  $\#V_i(\mathbb{F}_q) \leq \deg(V_i)q^{\dim V_i}$ .  $\square$

**Lemma 2.4.** *Let  $\mathbf{k}$  be any field and let  $X \subseteq \mathbb{A}_{\mathbf{k}}^r$ . Let  $f_1, \dots, f_t$  be polynomials in  $\mathbf{k}[x_1, \dots, x_r]$ . If  $X' = X \cap \mathbb{V}(f_1, \dots, f_t)$ , then  $\widehat{\deg}(X') \leq \widehat{\deg}(X) \cdot \prod_{i=1}^t \deg(f_i)$ .*

*Proof.* This follows from the refined version of Bezout's Theorem [Ful98, Example 12.3.1].  $\square$

### 3. A UNIFORM LOWER BOUND ON HILBERT FUNCTIONS

For a subscheme of  $\mathbb{P}^r$ , the Hilbert function in degree  $d$  is controlled by the Hilbert polynomial, at least if  $d$  is very large related to some invariants of the subscheme. We analyze the Hilbert function of a subscheme at the opposite extreme, where the degree of the subscheme is much larger than  $d$ . The following lemma, which applies to subschemes of arbitrarily high degree, provides uniform lower bounds that are crucial to bounding the error in our sieves.

**Lemma 3.1.** *Let  $\mathbf{k}$  be an arbitrary field and fix some  $e \geq 0$ . Let  $V \subseteq \mathbb{P}_{\mathbf{k}}^r$  be any closed,  $m$ -dimensional subscheme of degree  $> e$  with homogeneous coordinate ring  $R$ .*

(1) *We have  $\dim R_d \geq h^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d))$  for all  $d$ .*

(2) *For any  $0 < \epsilon < 1$ , there exists a constant  $C$  depending only on  $e, m$  and  $\epsilon$  such that*

$$\dim R_d > (e + \epsilon) \cdot h^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d))$$

*for all  $d \geq Ce^{m+1}$ .*

*Proof.* We can assume that we are working over an infinite field, since this will not change the values of the Hilbert function of  $R$ . For part (1), we simply take a linear Noether normalization  $\mathbf{k}[t_0, t_1, \dots, t_m] \subseteq R$  of the ring  $R$  [Eis95, Theorem 13.3]. This yields  $\mathbf{k}[t_0, t_1, \dots, t_m]_d \subseteq R_d$ , giving the statement about Hilbert functions.

We prove part (2) of the theorem by induction on  $m$ . Let  $S = \mathbf{k}[x_0, x_1, \dots, x_r]$  and let  $I_V \subseteq S$  be the saturated, homogeneous ideal defining  $V$ . Thus  $R = S/I_V$ . If  $m = 0$ , then we have  $\dim R_d \geq \min\{d+1, \deg V\} \geq \min\{d+1, e+1\}$  which is at least  $e + \epsilon$  for all  $d \geq e$ . This proves the case  $m = 0$ , where the constant  $C$  can be chosen to be 1.

Now assume the claim holds for all closed subschemes of dimension less than  $m$ . Let  $V \subset \mathbb{P}_{\mathbf{k}}^r$  be a closed subscheme with  $\dim V = m \geq 1$ . Fix  $0 < \epsilon < 1$ . Since we are working over an infinite field, [Eis95, Lemma 13.2(c)] allows us to choose a linear form  $\ell$  that is a nonzero divisor on  $R$ . Hence we have a short exact sequence:

$$(1) \quad 0 \longrightarrow R(-1) \xrightarrow{\cdot \ell} R \longrightarrow R/\ell \longrightarrow 0.$$

Letting  $W = V \cap V(\ell)$  we know that  $\dim W = m - 1$  and  $\deg W = \deg V$ . Moreover, If  $I_V$  is the saturated ideal defining  $V$  and if  $I_W$  is the saturated ideal defining  $W$ , then we have that  $I_W$  is the saturation of  $I_V + \langle \ell \rangle$  at  $\mathfrak{m}$ . In particular, since  $I_W$  contains  $I_V + \langle \ell \rangle$ , we have

$$\dim(S/(I_V + \langle \ell \rangle))_i \geq \dim(S/I_W)_i.$$

By induction, for  $\epsilon'$  with  $0 < \epsilon < \epsilon' < 1$ , there exists  $C'$  depending on  $\epsilon', e$  and  $m - 1$  where

$$\dim(S/I_W)_i \geq (e + \epsilon') \binom{m - 1 + i}{m - 1}$$

for all  $i \geq C'e^m$ . Iteratively applying the exact sequence (1) for  $d \geq C'e^m$  we obtain:

$$\begin{aligned} \dim R_d &\geq \dim R_{C'e^m} + \sum_{i=C'e^{m-1}}^d \dim(S/I_V + \ell)_i \\ &\geq \dim R_{C'e^m} + \sum_{i=C'e^{m-1}}^d \dim(S/I_W)_i \\ &\geq \sum_{i=C'e^m}^d (e + \epsilon') \binom{m - 1 + i}{m - 1}. \end{aligned}$$

The identity  $\sum_{i=a}^b \binom{i+k}{k} = \binom{b+k+1}{k+1} - \binom{a+k}{k+1}$  implies that  $\sum_{i=C'e^m}^d (e + \epsilon') \binom{m-1+i}{m-1}$  can be rewritten as  $(e + \epsilon') \left( \binom{m+d}{m} - \binom{m-1+C'e^m}{m} \right)$ . There exists a constant  $C$  depending on  $e, \epsilon$ , and  $m$  so that  $(\epsilon' - \epsilon) \binom{m+d}{m} \geq (e + \epsilon') \binom{m-1+C'e^m}{m}$  for all  $d \geq Ce^{m+1}$ . Thus, for all  $d \geq Ce^{m+1}$  we have

$$\dim R_d \geq (e + \epsilon') \binom{m+d}{m} - (\epsilon' - \epsilon) \binom{m+d}{d} = (e + \epsilon) \binom{m+d}{m}. \quad \square$$

*Remark 3.2.* Asymptotically in  $e$ , the bound of  $Ce^2$  is the best possible for curves. For instance, let  $C \subseteq \mathbb{P}^r$  be a curve of degree  $(e + 1)$  lying inside some plane  $\mathbb{P}^2 \subseteq \mathbb{P}^r$ . Let  $R$  be the homogeneous coordinate ring of  $C$ . If  $d \geq e$  then the Hilbert function is given by

$$\dim R_d = (e + 1)d - \frac{e^2 - e}{2}.$$

Thus, if we want  $\dim R_d \geq (e + \epsilon)(d + 1)$ , we will need to let  $d \geq \frac{e^2 + 3e + 2\epsilon}{2(1 - \epsilon)} \approx \frac{1}{2}e^2$ . It would be interesting to know if the bound  $Ce^{m+1}$  is the best possible for higher dimensional varieties.

#### 4. GEOMETRIC ANALYSIS

In this section we analyze the geometric picture for the distribution of parameters on  $X$ . The basic idea behind the proof of Theorem A is that  $f_0, f_1, \dots, f_k$  fail to be parameters on  $X$  if and only if they vanish along some  $(n - k)$ -dimensional subvariety of  $X$ . Since the Hilbert polynomial of any  $(n - k)$ -dimensional variety grows like  $d^{n-k}$ , when we restrict a degree  $d$  polynomial  $f_j$  to such a subvariety, it can be written in terms of  $\approx d^{n-k}$  distinct monomials. The polynomial  $f_j$  will vanish along the subvariety if and only if all of the  $\approx d^{n-k}$  coefficients vanish. This rough estimate explains the growth of the codimension of  $\mathcal{D}_{d,k}(X)$  as  $d \rightarrow \infty$ .

We begin by constructing the schemes  $\mathcal{D}_{k,d}(X)$ . Fix  $X \subseteq \mathbb{P}_{\mathbf{k}}^r$  a closed subscheme of dimension  $n$  over a field  $\mathbf{k}$ . Given  $k < n$  and  $d > 0$ , let  $\mathcal{A}_{k,d}$  be the affine space  $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))^{\oplus k+1}$  and



$\mathbf{k}[c_{0,1}, \dots, c_{k, \binom{r+d}{d}}]$  be the corresponding polynomial ring. We enumerate the monomials in  $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$  as  $m_1, \dots, m_{\binom{r+d}{d}}$ , and then define the universal polynomial

$$F_i := \sum_{j=1}^N c_{i,j} m_j \in \mathbf{k}[c_{0,1}, \dots, c_{k, \binom{r+d}{d}}] \otimes_{\mathbf{k}} \mathbf{k}[x_0, \dots, x_r].$$

Given a closed point  $c \in \mathcal{A}_{k,d}$  we can specialize  $F_0, \dots, F_k$  and obtain polynomials  $f_0, \dots, f_k \in \kappa(c)[x_0, \dots, x_r]$ , where  $\kappa(c)$  is the residue field of  $c$ . We will thus identify each element of  $\mathcal{A}_{k,d}(\mathbf{k})$  with a collection of polynomials  $\mathbf{f} = (f_0, f_1, \dots, f_k) \in \mathbf{k}[x_0, \dots, x_r]$ .

Now define  $\Sigma_{k,d}(X) \subseteq X \times \mathcal{A}_{k,d}$  via the equations  $F_0, \dots, F_k$ . Consider the second projection  $p_2 : \Sigma^{(k,d,X)} \rightarrow \mathcal{A}_{k,d}$ . Given a point  $\mathbf{f} = (f_0, \dots, f_k) \in \mathcal{A}_{k,d}$ , the fiber  $p_2^{-1}(\mathbf{f}) \subseteq X$  can be identified with the points lying in  $X \cap \mathbb{V}(f_0, \dots, f_k)$ . For generic choices of  $\mathbf{f}$  (after passing to an infinite field if necessary) the polynomials  $(f_0, \dots, f_k)$  will have codimension  $k+1$ , and thus the fiber  $p_2^{-1}(\mathbf{f})$  will have dimension  $n - k - 1$ .

There is a closed sublocus in  $\mathcal{D}_{k,d}(X) \subsetneq \mathcal{A}_{k,d}$  where the dimension of the fiber is at least  $n-k$ , and we give  $\mathcal{D}_{k,d}(X)$  the reduced scheme structure. It follows that  $\mathcal{D}_{k,d}(X)$  parametrizes collections  $\mathbf{f} = (f_0, \dots, f_k)$  of degree  $d$  polynomials which fail to be parameters on  $X$ .

*Remark 4.1.* If we fix  $X_{\mathbb{Z}} \subseteq \mathbb{P}_{\mathbb{Z}}^r$ , then we can follow the same construction to obtain a scheme  $\mathcal{D}_{k,d}(X_{\mathbb{Z}}) \subseteq \mathcal{A}_{k,d}$ . Writing  $X_{\mathbf{k}}$  as the pullback  $X \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbf{k}$ , we observe that the equations defining  $\Sigma_{k,d}(X_{\mathbf{k}})$  are obtained by pulling back the equations defining  $\Sigma_{k,d}(X_{\mathbb{Z}})$ . It follows that  $\mathcal{D}_{k,d}(X_{\mathbb{Z}}) \times_{\text{Spec } \mathbb{Z}} \text{Spec }(\mathbf{k})$  has the same set-theoretic support as  $\mathcal{D}_{k,d}(X_{\mathbf{k}})$ .

**Definition 4.2.** We let  $\mathcal{D}_{k,d}^{bad}(X)$  be the locus of points in  $\mathcal{D}_{k,d}(X)$  where  $f_0, \dots, f_{k-1}$  already fail to be parameters on  $X$  and let  $\mathcal{D}_{k,d}^{good}(X) := \mathcal{D}_{k,d}(X) \setminus \mathcal{D}_{k,d}^{bad}(X)$ . We set  $\mathcal{D}_{0,d}^{bad}(X) = \emptyset$ .

*Remark 4.3.* We have a splitting:

$$\begin{aligned} \mathcal{A}_{k,d} &\rightarrow \mathcal{A}_{k-1,d} \times \mathcal{A}_{0,d} \\ (f_0, \dots, f_k) &\mapsto ((f_0, \dots, f_{k-1}), f_k). \end{aligned}$$

Letting  $\pi : \mathcal{D}_{k,d}(X) \rightarrow \mathcal{A}_{k-1,d}$  be the induced projection, we will speak about the degree of the image of  $\pi$  (considered in  $\mathcal{A}_{k-1,d}$ ) and the degree of a fiber of  $\pi$  (considered in  $\mathcal{A}_{0,d}$ ).

*Proof of Theorem A.* First consider the case  $k = n$ . There is a natural rational map from  $\mathcal{A}_{n,d}$  to the Grassmanian  $\text{Gr}(n+1, S_d)$  given by sending the polynomials  $(f_0, \dots, f_n)$  to the linear space that they span. Inside of the Grassmanian, the locus of choices of  $(f_0, \dots, f_n)$  that all vanish on a point of  $X$  is a divisor in the Grassmanian defined by the Chow form; see [GKZ08, 3.2.B]. The preimage of this hypersurface in  $\mathcal{A}_{n,d}$  is a hypersurface contained in  $\mathcal{D}_{n,d}(X)$ , and thus  $\mathcal{D}_{n,d}(X)$  has codimension 1.

For  $k < n$ , we will induct on  $k$ . Let  $k = 0$ . A polynomial  $f_0$  will fail to be a parameter on  $X$  if and only if  $\dim X = \dim(X \cap \mathbb{V}(f_0))$ . This happens if and only if  $f_0$  is a zero divisor on a top-dimensional component of  $X$ . Let  $V$  be the reduced subscheme of some top-dimensional irreducible component of  $X$  and let  $\mathcal{I}_V$  be the defining ideal sheaf of  $V$ . Then the set of zero divisors of degree  $d$  on  $V$  will form a linear subspace in  $\mathcal{A}_{0,d}$  corresponding to the elements of the vector subspace  $H^0(\mathcal{I}_V(d))$ . The codimension of  $H^0(\mathcal{I}_V(d)) \subseteq S_d$  is precisely given

by the Hilbert function of the homogeneous coordinate ring of  $V$  in degree  $d$ . By applying Lemma 3.1(1), we conclude that for all  $d$  this linear space has codimension at least  $\binom{n+d}{d}$ . Since  $\mathcal{D}_{0,d}(X)$  is the union of these linear spaces over all top-dimensional components of  $X$ , this proves that  $\text{codim } \mathcal{D}_{0,d}(X) \geq \binom{n+d}{d}$ .

Take the induction hypothesis that we have proven the statement for  $\mathcal{D}_{j,d}(X')$  for all  $X' \subseteq \mathbb{P}^r$  and all  $j \leq k-1$ . We separate  $\mathcal{D}_{k,d}(X) = \mathcal{D}_{k,d}^{\text{bad}}(X) \sqcup \mathcal{D}_{k,d}^{\text{good}}(X)$  and will show that each locus has sufficiently large codimension. We begin with  $\mathcal{D}_{k,d}^{\text{bad}}(X)$ . By definition, the projection  $\pi$  from Remark 4.3 maps  $\mathcal{D}_{k,d}^{\text{bad}}(X)$  onto  $\mathcal{D}_{k-1,d}(X)$ . We thus have:

$$\text{codim}(\mathcal{D}_{k,d}^{\text{bad}}(X), \mathcal{A}_{k,d}) \geq \text{codim}(\mathcal{D}_{k-1,d}(X), \mathcal{A}_{k-1,d}) \geq \binom{n-k+1+d}{n-k+1} \geq \binom{n-k+d}{n-k},$$

where the middle inequality follows by induction.

Now consider an arbitrary point  $\mathbf{f} = (f_0, \dots, f_k)$  in  $\mathcal{D}_{k,d}^{\text{good}}(X)$ . By definition,  $f_0, \dots, f_{k-1}$  must be parameters on  $X$ , and thus  $\pi(\mathbf{f}) \in \mathcal{A}_{k-1,d} \setminus \mathcal{D}_{k-1,d}(X)$ . Using the splitting of Remark 4.3, the fiber of  $\mathcal{D}_{k,d}^{\text{good}}(X)$  over  $\mathbf{f}$  can be identified with  $\mathcal{D}_{0,d}(X')$  where  $X' := X \cap \mathbb{V}(f_0, \dots, f_{k-1})$ . Since  $(f_0, \dots, f_{k-1}) \notin \mathcal{D}_{k-1,d}(X)$ , we have that  $\dim X' = n - k$ . The inductive hypothesis thus guarantees that  $\text{codim } \mathcal{D}_{0,d}(X') \geq \binom{\dim X' + d}{d} = \binom{n-k+d}{d}$ .  $\square$

## 5. PROBABILISTIC ANALYSIS I: PROOF OF THEOREM B

Throughout this section, we let  $X \subseteq \mathbb{P}_{\mathbb{F}_q}^r$  be a projective scheme of dimension  $n$  over a finite field  $\mathbb{F}_q$ . Recall that  $S_d = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$ . We define

$$\text{Par}_{d,k} = \left\{ (f_0, f_1, \dots, f_k) \text{ that } \begin{array}{l} \text{are parameters on } X \end{array} \right\} \subset S_d^{k+1}.$$

In Theorem B, we compute the following limit (which a priori might not exist):

$$\lim_{d \rightarrow \infty} \text{Prob} \left( \begin{array}{l} (f_0, f_1, \dots, f_k) \text{ of degree } d \\ \text{are parameters on } X \end{array} \right) := \lim_{d \rightarrow \infty} \frac{\#\text{Par}_{d,k}}{\#S_d^{k+1}}.$$

As in the geometric case, there is a bifurcation between when  $k = n$  and  $k < n$ . The case  $k = n$  largely parallels Poonen's work [Poo04]: we will sieve over closed points of  $X$  and show that the asymptotic probability that  $(f_0, f_1, \dots, f_n)$  are parameters on  $X$  equals the product of local probabilities at the closed points of  $X$ , and the resulting formula will correspond with the value of a zeta function. This approach was already worked out by Bucur and Kedlaya in [BK12]. They assume that  $X$  is smooth, but their proof does not need that assumption.

When  $k < n$  we need to significantly alter Poonen's sieve. In this section, we focus on proving that the asymptotic probability converges to 1 as  $d \rightarrow \infty$ . For this, we will use a coarse error bound based on the geometric picture developed in §4. In §6, we provide a deeper analysis of the limit probability based on the detailed geometry of  $X$ .

**Proposition 5.1.** *If  $k < n$  then*

$$\text{Prob} \left( \begin{array}{l} (f_0, f_1, \dots, f_k) \text{ of degree } d \\ \text{are parameters on } X \end{array} \right) \geq 1 - \widehat{\deg}(X)(1 + d + d^2 + \dots + d^k)q^{-\binom{n-k+d}{n-k}}.$$



*Proof.* We induct on  $k$  and largely follow the structure of the proof of Theorem A. First, let  $k = 0$ . A polynomial  $f_0$  will fail to be a parameter on  $X$  if and only if it is a zero divisor on a top-dimensional component  $V$  of  $X$ . There are at most  $\widehat{\deg}(X)$  many such components. As argued in the proof of Theorem A, the set of zero divisors on  $V$  corresponds to the elements of  $H^0(\mathbb{P}^r, \mathcal{I}_V(d))$  which has codimension at least  $\binom{n+d}{d}$  in  $S_d$ . It follows that

$$\text{Prob} \left( \begin{array}{c} f_0 \text{ of degree } d \text{ is} \\ \text{not a parameter on } X \end{array} \right) \leq \widehat{\deg}(X) q^{-\binom{n+d}{d}}.$$

Now consider the induction step. We will separately compute the probability that  $\mathbf{f} = (f_0, f_1, \dots, f_k)$  lies in  $\mathcal{D}_{k,d}^{\text{bad}}(X)$  and the probability that  $\mathbf{f}$  lies in  $\mathcal{D}_{k,d}^{\text{good}}(X)$ . By definition, the projection  $\pi$  maps  $\mathcal{D}_{k,d}^{\text{bad}}(X)$  onto  $\mathcal{D}_{k-1,d}(X)$ , and by induction

$$\begin{aligned} \text{Prob}(\pi(\mathbf{f}) \in \mathcal{D}_{k-1,d}(X)(\mathbb{F}_q)) &\leq \widehat{\deg}(X) (1 + d + d^2 + \dots + d^{k-1}) q^{-\binom{n-k+1+d}{n-k+1}} \\ &\leq \widehat{\deg}(X) (1 + d + d^2 + \dots + d^{k-1}) q^{-\binom{n-k+d}{n-k}}. \end{aligned}$$

We now assume  $\mathbf{f} \notin \mathcal{D}_{k,d}^{\text{bad}}(X)$ . We thus have that  $f_0, \dots, f_{k-1}$  are parameters on  $X$ . As in the proof of Theorem A, the fiber  $\pi^{-1}(\mathbf{f})$  can be identified with  $\mathcal{D}_{0,d}(X')$  where  $X' := X \cap \mathbb{V}(f_0, f_1, \dots, f_{k-1})$ . By construction  $\dim X' = n - k$  and by Lemma 2.4,  $\widehat{\deg}(X') \leq \widehat{\deg}(X) \cdot d^k$ . Our inductive hypothesis thus implies that

$$\text{Prob} \left( \begin{array}{c} (f_0, \dots, f_k) \in \mathcal{D}_{k,d}(X)(\mathbb{F}_q) \text{ given} \\ \text{that } (f_0, \dots, f_{k-1}) \notin \mathcal{D}_{k-1,d}(X)(\mathbb{F}_q) \end{array} \right) \leq \widehat{\deg}(X') q^{-\binom{n-k+d}{n-k}} \leq \widehat{\deg}(X) \cdot d^k q^{-\binom{n-k+d}{n-k}}.$$

Combining the estimates for  $\mathcal{D}_{k,d}^{\text{bad}}(X)$  and  $\mathcal{D}_{k,d}^{\text{good}}(X)$  yields the proposition.  $\square$

*Proof of Theorem B.*

If  $k < n$ , then we apply Proposition 5.1 to obtain

$$\lim_{d \rightarrow \infty} \text{Prob} \left( \begin{array}{c} (f_0, \dots, f_k) \text{ of degree } d \\ \text{are parameters on } X \end{array} \right) \geq \lim_{d \rightarrow \infty} 1 - \widehat{\deg}(X) (d^0 + d^1 + \dots + d^k) q^{-\binom{n-k+d}{n-k}} = 1.$$

Now let  $k = n$ . For completeness, we summarize the proof of [BK12, Theorem 1.2]. We fix  $e$ , which will go to  $\infty$ , and separate the argument into low, medium, and high degree cases.

*Low degree argument.* For a zero dimensional subscheme  $Y$ , we have that  $S_d$  surjects on  $H^0(Y, \mathcal{O}_Y(d))$  when  $d \geq \deg Y - 1$  [Poo04, Lemma 2.1]. So if  $d > \deg P - 1$ , the probability that  $f_0, f_1, \dots, f_n$  all vanish at a closed point  $P \in X$  is  $1 - q^{-(n+1)\deg P}$ . If  $Y \subseteq X$  is the union of all points of degree  $\leq e$ , and if  $d \geq \deg Y - 1$ , then the surjection onto  $H^0(Y, \mathcal{O}_Y(d))$  implies that the probabilities at the points  $P \in Y$  behave independently. This yields:

$$\text{Prob} \left( \begin{array}{c} f_0, f_1, \dots, f_n \text{ of degree } d \text{ are} \\ \text{parameters on } X \text{ at all points} \\ P \in X \text{ where } \deg(P) \leq e \end{array} \right) = \prod_{\substack{P \in X \\ \deg(P) \leq e}} 1 - q^{-(n+1)\deg P}.$$

*Medium degree argument.* Our argument is nearly identical to [Poo04, Lemma 2.4], and covers all points whose degree lies in the range  $[e, \frac{d}{n+1}]$ . For any such point  $P \in X$ ,  $S_d$  surjects onto  $H^0(P, \mathcal{O}_P(d))$  and thus the probability that  $(f_0, f_1, \dots, f_n)$  all vanish at  $P$  is  $q^{-\ell(n+1)}$ . By [LW54],  $\#X(\mathbb{F}_{q^\ell}) \leq Kq^{\ell n}$  for some constant  $K$  independent of  $\ell$ . We have

$$\text{Prob} \left( \begin{array}{c} f_0, f_1, \dots, f_n \text{ of degree } d \text{ all} \\ \text{vanish at some } P \in X \\ \text{where } e < \deg(P) \leq \lfloor \frac{d}{n+1} \rfloor \end{array} \right) \leq \sum_{\ell=e}^{\lfloor \frac{d}{n+1} \rfloor} \#X(\mathbb{F}_{q^\ell}) q^{-\ell(n+1)} \leq \sum_{\ell=e}^{\infty} Kq^{\ell n} q^{-(n+1)\ell} = \frac{Kq^{-e}}{1 - q^{-1}}.$$

This tends to 0 as  $e \rightarrow \infty$ , and therefore does not contribute to the asymptotic limit.

*High degree argument.* By the case when  $k = n - 1$ , we may assume that  $f_0, f_1, \dots, f_{n-1}$  form a system of parameters with probability  $1 - o(1)$ . So we let  $V$  be one of the irreducible components of this intersection (over  $\mathbb{F}_q$ ) and we let  $R$  be its homogeneous coordinate ring. If  $\deg V \leq \frac{d}{n+1}$ , then it can be ignored as we considered such points in the low and medium degree cases. Hence, we can assume  $\deg V > \frac{d}{n+1}$ . Since  $\dim R_\ell \geq \min\{\ell + 1, \deg R\}$  for all  $\ell$ , the probability that  $f_n$  vanishes along  $V$  is at most  $q^{-\lfloor \frac{d}{n+1} \rfloor - 1}$ . Hence the probability of vanishing on some high degree point is bounded by  $O(d^n q^{-\lfloor \frac{d}{n+1} \rfloor - 1})$  which is  $o(1)$  as  $d \rightarrow \infty$ .

Combining the various parts as  $e \rightarrow \infty$ , we see that the low degree argument converges to  $\zeta_X(n+1)^{-1}$  and the contributions from the medium and high degree points go to 0.  $\square$

*Remark 5.2.* It might be interesting to consider variants of Theorem B that allow imposing conditions along closed subschemes, similar to Poonen's Bertini with Taylor Coefficients [Poo04, Theorem 1.2]. For instance, [Ked05, Theorem 1] might be provable by such an approach, though this would be more complicated than the original proof.

Proposition 5.1 also yields an effective bound on the degree of a full system of parameters over a finite field. Sharper bounds can be obtained if one allows the  $f_i$  to have different degrees.

### Corollary 5.3.

- (1) If  $d_1$  satisfies  $d_1^{n-1} q^{-d_1-1} < (n \cdot \widehat{\deg}(X))^{-1}$ , then there exist  $g_0, g_1, \dots, g_{n-1}$  of degree  $d_1$  that are parameters on  $X$ .
- (2) Let  $X'$  be 0-dimensional. If  $\max\{d_2 + 1, q\} \geq \widehat{\deg}(X')$  then there exists a degree  $d_2$  parameter on  $X'$ .

*Proof.* Applying Proposition 5.1 in the case  $k = n - 1$  yields (1). For (2), let  $f$  be a random degree  $d$  polynomial and let  $P \in X'$  be a closed point. Since the dimension of the image of  $S_d$  in  $H^0(P, \mathcal{O}_P(d))$  is at least  $\min\{d+1, \deg P\}$ , the probability that  $f$  vanishes at  $P$  is at worst  $q^{-\min\{d+1, \deg P\}}$  which is at least  $q^{-1}$ . It follows that the probability that a degree  $d$  function vanishes on some point of  $X'$  is at worst  $\sum_{P \in X'} q^{-1} \leq \widehat{\deg}(X') q^{-1}$ . Thus if  $q > \widehat{\deg}(X')$ , this happens with probability strictly less than 1. On the other hand, if  $d+1 \geq \widehat{\deg}(X')$  then polynomials of degree  $d$  surject onto  $H^0(X', \mathcal{O}_{X'}(d))$  and hence we can find a parameter on  $X'$  by choosing a polynomial that restricts to a unit on  $X'$ .  $\square$

*Proof of Corollary 1.2.* If  $\dim X = 0$ , then we can directly apply Corollary 5.3(2) to find a parameter of degree  $d = \max\{d, d^n\}$ . So we assume  $n := \dim X > 0$ . Since  $d > \log_q \widehat{\deg}(X) + \log_q n + n \log_q d$  it follows that  $(n \cdot \widehat{\deg}(X))^{-1} > q^{-d} d^n > q^{-d-1} d^{n-1}$ . Applying Corollary 5.3(1), we find  $g_0, g_1, \dots, g_{n-1}$  in degree  $d$  that are parameters on  $X$ . Let  $X' = X \cap V(g_0, g_1, \dots, g_{n-1})$ . Since  $\max\{d, \frac{q}{d^{n-1}}\} \geq \widehat{\deg}(X)$  it follows that  $\max\{d^n, q\} \geq d^{n-1} \widehat{\deg}(X) \geq \widehat{\deg}(X')$ , and Corollary 5.3(2) yields a parameter  $g_n$  of degree  $d^n$  on  $X'$ . Thus  $g_0^{d^n-d}, g_1^{d^n-d}, \dots, g_{n-1}^{d^n-d}, g_n$  are parameters of degree  $d^n = \max\{d, d^n\}$  on  $X$ .  $\square$

## 6. PROBABILISTIC ANALYSIS II: THE ERROR TERM

In this section, we let  $k < n$  and we analyze the error terms in Theorem B more precisely. In particular, we will show that the probabilities are controlled by the probability of vanishing along an  $(n - k)$ -dimensional subvariety, with varieties of lowest degree contributing the most. Theorem 6.2 is the estimate obtained by tracking subvarieties of degree  $\leq e$ .

**Notation 6.1.** For a subscheme  $Z \subseteq X$ , we write  $|Z|$  for the number of irreducible components of  $Z$ , and we write  $\dim Z \equiv k$  if  $Z$  is equidimensional of dimension  $k$ .

**Theorem 6.2.** Let  $X \subseteq \mathbb{P}_{\mathbb{F}_q}^r$  be a projective scheme of dimension  $n$ . Fix  $e$  and let  $k < n$ . The probability that random polynomials  $f_0, f_1, \dots, f_k$  of degree  $d$  are parameters on  $X$  is

$$\text{Prob} \left( \begin{array}{c} f_0, f_1, \dots, f_k \text{ of degree } d \\ \text{are parameters on } X \end{array} \right) = 1 - \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))} + o \left( q^{-e(k+1)\binom{n-k+d}{n-k}} \right).$$

The terms of the above sum have the form  $q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))}$ , where  $Z \subseteq X$  is an  $(n - k)$ -dimensional subvariety. Hence, the exponents are controlled by the Hilbert polynomial of a  $(n - k)$ -dimensional variety, and will grow like  $d^{n-k}$ , converging to 0 rapidly as  $d \rightarrow \infty$ .

Our proof of Theorem 6.2 adapts Poonen's sieve in a couple of key ways. The first big difference is that instead of sieving over closed points, we will sieve over  $(n - k)$ -dimensional subvarieties of  $X$ ; this is because polynomials  $(f_0, \dots, f_k)$  will fail to be parameters on  $X$  only if they vanish along some  $(n - k)$ -dimensional subvariety.

The second difference is that the resulting probability formula will not be a product of local factors. This is because the values of a function can never be totally independent along two higher dimensional varieties with a nontrivial intersection. For instance, Lemma 6.3 shows that the probability that a degree  $d$  polynomial vanishes along a line is  $q^{-(d+1)}$ , but the probability of vanishing along two lines that intersect in a point is  $q^{-(2d+1)} > (q^{-(d+1)})^2$ .

The following result characterizes the individual probabilities arising in our sieve.

**Lemma 6.3.** If  $Z \subseteq \mathbb{P}_{\mathbb{F}_q}^r$  is a reduced, projective scheme over a finite field  $\mathbb{F}_q$  with homogeneous coordinate ring  $R$  then

$$\text{Prob} \left( \begin{array}{c} (f_0, \dots, f_k) \text{ of degree } d \\ \text{vanish along } Z \end{array} \right) = \left( \frac{1}{\#R_d} \right)^{k+1}.$$

If  $d$  is at least the Castelnuovo-Mumford regularity of the ideal sheaf of  $Z$ , then

$$\text{Prob} \left( \begin{array}{c} (f_0, \dots, f_k) \text{ of degree } d \\ \text{vanish along } Z \end{array} \right) = q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))}.$$

*Proof.* Let  $I \subseteq S$  be the homogeneous ideal defining  $Z$ , so that  $R = S/I$ . An element  $h \in S_d$  vanishes along  $Z$  if and only if it restricts to 0 in  $R_d$  i.e. if and only if it lies in  $I_d$ . Since we have an exact sequence of  $\mathbb{F}_q$ -vector spaces:

$$0 \longrightarrow I_d \longrightarrow S_d \longrightarrow R_d \longrightarrow 0$$

we obtain

$$\text{Prob}(h \text{ vanishes on } Z) = \frac{\#I_d}{\#S_d} = \frac{1}{\#R_d}.$$

For  $k+1$  elements of  $S_d$ , the probabilities of vanishing along  $Z$  are independent and this yields the first statement of the lemma.

We write  $\tilde{I}$  for the ideal sheaf of  $Z$ . If  $d$  is at least the regularity  $\tilde{I}$ , then  $H^1(\mathbb{P}_{\mathbb{F}_q}^r, \tilde{I}(d)) = 0$ . Hence there is a natural isomorphism between  $R_d$  and  $H^0(Z, \mathcal{O}_Z(d))$ . Thus, we have

$$\frac{1}{\#R_d} = q^{-h^0(Z, \mathcal{O}_Z(d))},$$

yielding the second statement. □

*Proof of Theorem 6.2.* Throughout the proof, we set  $\epsilon_{e,k}$  to be the error term for a given  $e$  and  $k$ , namely  $\epsilon_{e,k} := q^{-e(k+1)\binom{n-k+d}{n-k}}$ . We also set:

$$\begin{aligned} \text{Par}_{d,k} &:= \left\{ \begin{array}{c} (f_0, f_1, \dots, f_k) \\ \text{are parameters on } X \end{array} \right\} \\ \text{Low}_{d,k,e} &:= \left\{ \begin{array}{c} (f_0, f_1, \dots, f_k) \text{ vanish along a variety } Z \\ \text{where } \dim Z = (n-k) \text{ and } \deg(Z) \leq e \end{array} \right\} \\ \text{Med}_{d,k,e} &:= \left\{ \begin{array}{c} (f_0, f_1, \dots, f_k) \notin \text{Low}_{d,k,e} \text{ which vanish along a variety } Z \\ \text{where } \dim Z = (n-k) \text{ and } e < \deg(Z) \leq e(k+1) \end{array} \right\} \\ \text{High}_{d,k,e} &:= \left\{ \begin{array}{c} (f_0, f_1, \dots, f_k) \notin \text{Low}_{d,k,e} \cup \text{Med}_{d,k,e} \text{ which vanish along a variety } Z \\ \text{where } \dim Z = (n-k) \text{ and } e(k+1) < \deg(Z) \end{array} \right\}. \end{aligned}$$

Note that if  $(f_0, f_1, \dots, f_k)$  vanish along a variety of dimension  $> n-k$  then they will also vanish along a high degree variety, and hence we do not need to count this case separately. For  $\mathbf{f} = (f_0, f_1, \dots, f_k) \in S_d^{k+1}$ , we thus have

$$\begin{aligned} \text{Prob}(\mathbf{f} \in \text{Par}_{d,k}) &= 1 - \text{Prob}(\mathbf{f} \in \text{Low}_{d,k,e} \cup \text{Med}_{d,k,e} \cup \text{High}_{d,k,e}) \\ &= 1 - \text{Prob}(\mathbf{f} \in \text{Low}_{d,k,e}) - \text{Prob}(\mathbf{f} \in \text{Med}_{d,k,e}) - \text{Prob}(\mathbf{f} \in \text{High}_{d,k,e}). \end{aligned}$$

It thus suffices to show that

$$\text{Prob}(\mathbf{f} \in \text{Low}_{d,k,e}) = \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))} + o(\epsilon_{e,k})$$

and that  $\text{Prob}(\text{Med}_{d,k,e})$  and  $\text{Prob}(\text{High}_{d,k,e})$  are each in  $o(\epsilon_{e,k})$ .

We proceed by induction on  $k$ . When  $k = 0$  the condition that  $f_0$  is a parameter on  $X$  is equivalent to  $f_0$  not vanishing along a top-dimensional component of  $X$ . Thus, combining Lemma 6.3 with an inclusion/exclusion argument implies the exact result:

$$\text{Prob}(f_0 \in \text{Par}_{d,0}) = 1 - \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv n-k}} (-1)^{|Z|-1} q^{-h^0(Z, \mathcal{O}_Z(d))}.$$

By basic properties of the Hilbert polynomial, as  $d \rightarrow \infty$  we have

$$h^0(Z, \mathcal{O}_Z(d)) = \frac{\deg(Z)}{n!} d^n + o(d^n) = \deg(Z) \binom{n+d}{d} + o(d^n).$$

Hence for the fixed degree bound  $e$ , we obtain:

$$\begin{aligned} \text{Prob}(\text{Par}_{d,0}) &= 1 - \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-h^0(Z, \mathcal{O}_Z(d))} - \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv n-k \\ \deg Z > e}} (-1)^{|Z|-1} q^{-h^0(Z, \mathcal{O}_Z(d))} \\ &= 1 - \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-h^0(Z, \mathcal{O}_Z(d))} + o(\epsilon_{e,0}). \end{aligned}$$

We now consider the induction step. Let  $\mathbf{f} = (f_0, \dots, f_k)$  drawn randomly from  $S_d^{k+1}$ . Here we separate into low, medium, and high degree cases.

*Low degree argument.* Let  $\mathbf{V}_{k,e}$  denote the set of integral projective varieties  $V \subseteq X$  of dimension  $n - k$  and degree  $\leq e$ . We have  $\mathbf{f} \in \text{Low}_{d,k,e}$  if and only if  $\mathbf{f}$  vanishes on some  $V \in \mathbf{V}_{k,e}$ . Since  $\mathbf{V}_{k,e}$  is a finite set, we may use an inclusion-exclusion argument to get

$$\text{Prob}(\mathbf{f} \in \text{Low}_{d,k,e}) = \sum_{\substack{Z \subseteq X \text{ a union of} \\ V \in \mathbf{V}_{k,e}}} (-1)^{|Z|-1} \text{Prob} \left( \begin{array}{c} f_0, \dots, f_k \text{ of degree } d \\ \text{vanish along } Z \end{array} \right).$$

If  $\deg Z > e$  then Lemma 6.3 implies that those terms can be absorbed into the error term  $o(\epsilon_{e,k})$ . Moreover, assuming that  $Z$  is a union of  $V \in \mathbf{V}_{k,e}$  satisfying  $\deg(Z) \leq e$  is equivalent to assuming  $Z$  is reduced and equidimensional of dimension  $n - k$ . We thus have:

$$= \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} \text{Prob} \left( \begin{array}{c} f_0, \dots, f_k \text{ of degree } d \\ \text{vanish along } Z \end{array} \right) + o(\epsilon_{e,k}).$$

*Medium degree argument.* We know that  $\text{Prob}(\mathbf{f} \in \text{Med}_{d,k,e})$  is bounded by the sum of the probabilities that  $f$  vanishes along some irreducible variety  $V$  in  $\mathbf{V}_{e(k+1),k} \setminus \mathbf{V}_{e,k}$ .

$$\text{Prob}(\mathbf{f} \in \text{Med}_{d,k,e}) \leq \sum_{Z \in \mathbf{V}_{e(k+1),k} \setminus \mathbf{V}_{e,k}} \text{Prob} \left( \begin{array}{c} (f_0, f_1, \dots, f_k) \text{ of degree } d \\ \text{vanish along } Z \end{array} \right).$$

Lemma 6.3 implies that each summand on the right-hand side lies in  $o(\epsilon_{e,k})$ . This sum is finite and thus  $\text{Prob}(\mathbf{f} \in \text{Med}_{d,k,e})$  is in  $o(\epsilon_k)$ .

*High degree argument.* Proposition 5.1 implies that  $f_0, f_1, \dots, f_{k-1}$  are parameters on  $X$  with probability  $1 - o\left(q^{-\binom{n-k+1+d}{d}}\right) \geq 1 - o(\epsilon_{e,k})$  for any  $e$ . Hence we may restrict our attention to the case where  $f_0, \dots, f_{k-1}$  are parameters on  $X$ .

Let  $V_1, V_2, \dots, V_s$  be the irreducible components of  $X' := X \cap \mathbb{V}(f_0, f_1, \dots, f_{k-1})$  that have dimension  $n - k$ . We have that  $f_0, f_1, \dots, f_k$  fail to be parameters on  $X$  if and only if  $f_k$  vanishes on some  $V_i$ . We can assume that  $f_k$  does not vanish on any  $V_i$  where  $\deg V_i \leq e(k+1)$  as we have already accounted for this possibility in the low and medium degree cases. After possibly relabelling the components, we let  $V_1, V_2, \dots, V_t$  be the components of degree  $> e(k+1)$  and  $X'' = V_1 \cup V_2 \cup \dots \cup V_t$ . Using Lemma 2.4, we compute  $\widehat{\deg}(X'') \leq \widehat{\deg}(X') = \widehat{\deg}(X) \cdot d^k$ . It follows that  $X''$  has at most  $\frac{\widehat{\deg}(X)d^k}{e(k+1)}$  irreducible components.

Since the value of  $d$  is not necessarily larger than the Castelnuovo-Mumford regularity of  $V_i$ , we cannot use a Hilbert polynomial computation to bound the probability that  $f_k$  vanishes along  $V_i$ . Instead, we use the lower bound for Hilbert functions obtained in Lemma 3.1. Let  $\epsilon = 1/2$ , though any choice of  $\epsilon$  would work. We write  $R(V_i)$  for homogeneous coordinate ring of  $V_i$ . For any  $1 \leq i \leq t$ , Lemmas 3.1 and 6.3 yield

$$\text{Prob} \left( \begin{array}{c} f_k \text{ of degree } d \\ \text{vanishes along } V_i \end{array} \right) = q^{-\dim R(V_i)_d} \leq q^{-(e(k+1)+\epsilon)\binom{n-k+d}{n-k}}$$

whenever  $d \geq Ce^{k+1}$ . Combining this with our bound on the number of irreducible components of  $X''$  gives  $\text{Prob}(\mathbf{f} \in \text{High}_{d,k,e}) \leq \frac{\widehat{\deg} X}{e(k+1)} d^k q^{-(e(k+1)+\epsilon)\binom{n-k+d}{n-k}}$  which is in  $o(\epsilon_{e,k})$ .  $\square$

*Proof of Corollary 1.1.* Let  $N$  denote the number of  $(n-k)$ -planes spaces  $L \subseteq \mathbb{P}_{\mathbb{F}_q}^r$  such that  $L \subseteq X$ . Choosing  $e = 1$  in Theorem 6.2, we compute that

$$\text{Prob} \left( \begin{array}{c} f_0, f_1, \dots, f_k \text{ of degree } d \\ \text{are parameters on } X \end{array} \right) = 1 - Nq^{-(k+1)\binom{n-k+d}{n-k}} + o\left(q^{-(k+1)\binom{n-k+d}{n-k}}\right).$$

It follows that

$$\text{Prob} \left( \begin{array}{c} f_0, f_1, \dots, f_k \text{ of degree } d \\ \text{are not parameters on } X \end{array} \right) = Nq^{-(k+1)\binom{n-k+d}{n-k}} + o\left(q^{-(k+1)\binom{n-k+d}{n-k}}\right).$$

Dividing both sides by  $q^{-(k+1)\binom{n-k+d}{n-k}}$  and taking the limit as  $d \rightarrow \infty$  yields the corollary.  $\square$



## 7. PROOFS OVER $\mathbb{Z}$ AND $\mathbb{F}_q[t]$

In this section we prove Theorem C and Corollary 1.3.

**Definition 7.1.** Let  $B = \mathbb{Z}$  or  $\mathbb{F}_q[t]$  and fix a finitely generated, free  $B$ -module  $B^s$  and a subset  $\mathcal{S} \subseteq B^s$ . Given  $a \in B^s$  we write  $a = (a_1, a_2, \dots, a_s)$ . The **density** of  $\mathcal{S} \subseteq B$  is

$$\text{Density}(\mathcal{S}) := \begin{cases} \lim_{N \rightarrow \infty} \frac{\#\{a \in \mathcal{S} \mid \max\{|a_i|\} \leq N\}}{\#\{a \in \mathbb{Z}^s \mid \max\{|a_i|\} \leq N\}} & \text{if } B = \mathbb{Z} \\ \lim_{N \rightarrow \infty} \frac{\#\{a \in \mathcal{S} \mid \max\{\deg a_i\} \leq N\}}{\#\{a \in \mathbb{F}_q[t]^s \mid \max\{\deg a_i\} \leq N\}} & \text{if } B = \mathbb{F}_q[t] \end{cases}$$

*Proof of Theorem C.* For clarity, we will prove the result over  $\mathbb{Z}$  in detail and at the end, mention the necessary adaptations for  $\mathbb{F}_q[t]$ .

We first let  $k < n$ . Given degree  $d$  polynomials  $(f_0, f_1, \dots, f_k)$  with integer coefficients and a prime  $p$ , let  $(\overline{f}_0, \overline{f}_1, \dots, \overline{f}_k)$  be the reduction of these polynomials mod  $p$ . Then  $(\overline{f}_0, \overline{f}_1, \dots, \overline{f}_k)$  will be parameters on  $X_p$  if and only if the point  $\overline{\mathbf{f}} = (\overline{f}_0, \overline{f}_1, \dots, \overline{f}_k)$  lies  $\mathcal{D}_{d,k}(X_{\mathbb{F}_p})$ . As noted in Remark 4.1, this is equivalent to asking that  $\overline{\mathbf{f}}$  is an  $\mathbb{F}_p$ -point of  $\mathcal{D}_{k,d}(X_{\mathbb{Z}})$ . Thus, we may apply [Eke91, Theorem 1.2] to  $\mathcal{D}_{d,k}(X_{\mathbb{Z}}) \subseteq \mathcal{A}_{k,d}$  (using  $M = 1$ ) to conclude that

$$\text{Density} \left\{ \begin{array}{l} (f_0, \dots, f_k) \text{ of degree } d \text{ that restrict} \\ \text{to parameters on } X_p \text{ for all } p \end{array} \right\} = \prod_p \text{Prob} \left( \begin{array}{l} (f_0, \dots, f_k) \text{ of degree } d \\ \text{restrict to parameters on } X_p \end{array} \right).$$

Applying Proposition 5.1 to estimate the individual factors; we have:

$$\begin{aligned} \text{Density} \left\{ \begin{array}{l} (f_0, \dots, f_k) \text{ of degree } d \text{ that restrict} \\ \text{to parameters on } X_p \text{ for all } p \end{array} \right\} &= \lim_{d \rightarrow \infty} \prod_p \text{Prob} \left( \begin{array}{l} (f_0, \dots, f_k) \text{ of degree } d \\ \text{restrict to parameters on } X_p \end{array} \right) \\ &\geq \lim_{d \rightarrow \infty} \prod_p \left( 1 - \widehat{\deg}(X_p)(1 + d + \dots + d^k)p^{-\binom{n-k+d}{n-k}} \right). \end{aligned}$$

Lemma 7.2 shows that there is an integer  $D$  where  $D \geq \widehat{\deg}(X_p)$  for all  $p$ . Moreover,  $1 + d + \dots + d^k \leq kd^k$  for all  $d$ , and hence:

$$\geq \lim_{d \rightarrow \infty} \prod_p \left( 1 - Dkd^k p^{-\binom{n-k+d}{n-k}} \right).$$

For  $d \gg 0$  we can make  $Dkd^k p^{-\binom{n-k+d}{n-k}} \leq p^{-d/2}$  for all  $p$  simultaneously. Using  $\zeta(n)$  for the Riemann-Zeta function, we get:

$$\geq \lim_{d \rightarrow \infty} \prod_p (1 - p^{-d/2}) \geq \lim_{d \rightarrow \infty} \zeta(d/2)^{-1} = 1.$$

We now consider the case  $k = n$ . This follows by a “low degree argument” exactly analogous to [Poo04, Theorem 5.13]. Fix a large integer  $N$  and let  $Y$  be the union of all closed points

$P \in X$  whose residue field  $\kappa(P)$  has cardinality at most  $N$ . Since  $Y$  is a finite union of closed, we see that for  $d \gg 0$ , there is a surjection:

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)) \longrightarrow H^0(Y, \mathcal{O}_Y(d)) \cong \bigoplus_{\substack{P \in X \\ \#\kappa(P) \leq N}} H^0(P, \mathcal{O}_P(d)) \longrightarrow 0.$$

It follows that we have a product formula:

$$\text{Density} \left\{ (f_0, f_1, \dots, f_n) \text{ of degree } d \text{ do not vanish} \right. \\ \left. \text{on a point } P \text{ with } \#\kappa(P) \leq N \right\} = \prod_{P \in X, \#\kappa(P) \leq N} \left( 1 - \frac{1}{\#\kappa(P)^{n+1}} \right)$$

This is certainly an upper bound on the density of  $(f_0, f_1, \dots, f_n)$  that are parameters on  $X_p$  for all  $p$ . As  $N \rightarrow \infty$  the righthand side approaches  $\zeta_X(n+1)^{-1}$ . However, since the dimension of  $X$  is  $n+1$ , this zeta function has a pole at  $s = n+1$  [Ser65, Theorems 1 and 3(a)]. Hence this asymptotic density equals 0. This completes the proof over  $\mathbb{Z}$ .

Over  $\mathbb{F}_q[t]$ , the key adaptation is to use [Poo03, Theorem 3.1] in place of Ekedahl's result. Poonen's result is stated for a pair of polynomials, but it applies equally well to  $n$ -tuples of polynomials such as the  $n$ -tuples defining  $\mathcal{D}_{k,d}(X)$ . In particular, one immediately reduces to proving an analogue of [Poo03, Lemma 5.1], for  $n$ -tuples of polynomials which are irreducible over  $\mathbb{F}_q(t)$  and which have gcd equal to 1; but the  $n = 2$  version of the lemma then implies the  $n \geq 2$  versions of the lemma.<sup>1</sup> The rest of our argument over  $\mathbb{Z}$  works over  $\mathbb{F}_q[t]$ .  $\square$

**Lemma 7.2.** *Let  $X \subseteq \mathbb{P}_B^r$  be any closed subscheme. There is an integer  $D$  where  $D \geq \widehat{\deg}(X_s)$  for all  $s \in \text{Spec } B$ .*

*Proof.* First we take a flattening stratification for  $X$  over  $B$  [GD67, Corollaire 6.9.3]. Within each strata, the maximal degree of a minimal generator is semicontinuous, and we can thus find a degree  $e$  where  $X_s$  is generated in degree  $e$  for all  $s \in \text{Spec } B$ . By [BM93, Prop. 3.5], we then obtain that  $\widehat{\deg}(X) \leq \sum_{j=0}^n e^{r-j}$ . In particular defining  $D := re^r$  will suffice.  $\square$

To prove Corollary 1.3, we use Theorem C to find a submaximal collection  $(f_0, f_1, \dots, f_{n-1})$  which restrict to parameters on  $X_s$  for all  $s \in \text{Spec } B$ . This cuts  $X$  down to a scheme  $X' = X \cap \mathbb{V}(f_0, f_1, \dots, f_{n-1})$  with 0-dimensional fibers over each point  $s$ . When  $B = \mathbb{Z}$ , such a scheme is essentially a union of orders in number fields, and we find the last element  $f_n$  by applying classical arithmetic results about the Picard groups of rings of integers of number fields. When  $B = \mathbb{F}_q[t]$ , we use similar facts about Picard groups of affine curves over  $\mathbb{F}_q$ .

An example illustrates this approach. Let  $X = \mathbb{P}_{\mathbb{Z}}^1 = \text{Proj}(\mathbb{Z}[x, y])$ . A polynomial of degree  $d$  will be a parameter on  $X$  as long as the  $d+1$  coefficients are relatively prime. Thus as  $d \rightarrow \infty$ , the density of these choices will go to 1. However, once we have fixed one such parameter, say  $5x - 3y$ , it is much harder to find an element that will restrict to a parameter on  $\mathbb{Z}[x, y]/(5x - 3y)$  modulo  $p$  for all  $p$ . In fact, the only possible choices are the elements which restrict to units on  $\text{Proj}(\mathbb{Z}[x, y]/(5x - 3y))$ . Among the linear forms, these are

$$\pm(7x - 4y) + c(5x - 3y) \text{ for any } c \in \mathbb{Z}.$$

Hence, these elements arise with density zero, and yet they form a nonempty subset.

**Lemma 7.3.** *If  $X' \subseteq \mathbb{P}_{\mathbb{Z}}^r$  is closed and finite over  $\text{Spec}(\mathbb{Z})$ , then  $\text{Pic}(X')$  is finite.*

<sup>1</sup>We thank Bjorn Poonen for pointing out this reduction.

*Proof.* We first reduce to the case where  $X'$  is reduced. Let  $\mathcal{N} \subseteq \mathcal{O}_{X'}$  be the nilradical ideal. If  $X$  is nonreduced then there is some integer  $m > 1$  for which  $\mathcal{N}^m = 0$ . Let  $X''$  be the closed subscheme defined by  $\mathcal{N}^{m-1}$ . We have a short exact sequence

$$0 \longrightarrow \mathcal{N}^{m-1} \longrightarrow \mathcal{O}_{X'}^* \longrightarrow \mathcal{O}_{X''}^* \longrightarrow 1$$

where the first map sends  $f \mapsto 1 + f$ . Since  $H^1(X', \mathcal{N}^{m-1}) = H^2(X', \mathcal{N}^{m-1}) = 0$ , taking cohomology yields an isomorphism  $\text{Pic}(X') \cong \text{Pic}(X'')$ . Iterating this argument, we may assume  $X'$  is reduced.

We now have  $X' = \text{Spec}(B)$  where  $B$  is a finite, reduced  $\mathbb{Z}$ -algebra. If  $Q$  is a minimal prime of  $B$ , then  $B/Q$  is either zero dimensional or an order in a number field, and hence has a finite Picard group [Neu99, Theorem 12.12]. If  $Q'$  is the intersection of all of the other minimal primes  $B$ . Then we again have an exact sequence in cohomology

$$\dots \longrightarrow (B/(Q + Q'))^* \longrightarrow \text{Pic}(X') \longrightarrow \text{Pic}(B/Q) \oplus \text{Pic}(B/Q') \longrightarrow \dots$$

Since  $(B/(Q + Q'))^*$  is a finite set, and since  $B/Q$  and  $B/Q'$  have fewer minimal primes than  $B$ , we may use induction to conclude that  $\text{Pic}(X')$  is finite.  $\square$

**Lemma 7.4.** *If  $C$  is an affine curve over  $\mathbb{F}_q$ , then  $\text{Pic}(C)$  is finite.*

*Proof.* If  $C$  fails to be integral, then an argument entirely analogous to the proof of Lemma 7.3 reduces us to the case  $C$  is integral. We next assume that  $C$  is nonsingular and integral, and that  $\overline{C}$  is the corresponding nonsingular projective curve. Since  $C$  is affine we have  $\text{Pic}(C) = \text{Pic}^0(C) \subseteq \text{Pic}^0(\overline{C}) \subseteq \text{Jac}(\overline{C})(\mathbb{F}_q)$ , which is a finite set. If  $C$  is singular, then the finiteness of  $\text{Pic}(C)$  follows from the nonsingular case by a minor adaptation of the proof of [Neu99, Proposition 12.9].  $\square$

*Proof of Corollary 1.3.* By Theorem C, for  $d \gg 0$  we can find polynomials  $f_0, f_1, \dots, f_{n-1}$  of degree  $d$  that restrict to parameters on  $X_s$  for all  $s \in \text{Spec } B$ . Let  $X' := \mathbb{V}(f_0, f_1, \dots, f_{n-1}) \cap X$ , which is finite over  $B$  by construction. Let  $A$  be the finite  $B$ -algebra where  $\text{Spec } A = X'$ . Lemma 7.3 or 7.4 implies that  $H^0(X', \mathcal{O}_{X'}(e)) = A$  for some  $e$ . We can thus find a polynomial  $f_n$  of degree  $e$  mapping onto a unit in the  $B$ -algebra  $A$ . It follows that  $\mathbb{V}(f_n) \cap X' = \emptyset$ . Replace  $f_i$  by  $f_i^e$  for  $i = 0, \dots, n-1$  and replace  $f_n$  by  $f_n^d$ . Then we have  $f_0, f_1, \dots, f_n$  of degree  $d' := de$  and restricting to parameters on  $X_s$  for all  $s \in \text{Spec}(B)$  simultaneously.

We thus obtain a proper morphism  $\pi : X \rightarrow \mathbb{P}_B^n$  where  $X_s \rightarrow \mathbb{P}_{\kappa(s)}^n$  is finite for all  $s$ . Since  $\pi$  is quasi-finite and proper, it is finite by [GD66, Théorème 8.11.1].  $\square$

The following generalizes Corollary 1.3 to other graded rings.

**Corollary 7.5.** *Let  $B = \mathbb{Z}$  or  $\mathbb{F}_q[t]$  and let  $R$  be a graded, finite type  $B$ -algebra where  $\dim R \otimes_{\mathbb{Z}} \mathbb{F}_p = n + 1$  for all  $p$ . Then there exist  $f_0, f_1, \dots, f_n$  of degree  $d$  for some  $d$  such that  $B[f_0, f_1, \dots, f_n] \subseteq R$  is a finite extension.*

*Proof.* After replacing  $R$  by a high degree Veronese subring  $R'$ , we may assume that  $R'$  is generated in degree one and contains no  $R'_+$ -torsion submodule, where  $R'_+ \subseteq R'$  is the homogeneous ideal of strictly positive degree elements. Let  $r + 1$  be the number of generators

of  $R'_1$ . Then there is a surjection  $\phi: B[x_0, \dots, x_r] \rightarrow R'$  inducing an embedding of  $X := \text{Proj}(R') \subseteq \mathbb{P}_B^r$ . Since  $R'$  contains no  $R'_+$ -torsion submodule, the kernel of  $\phi$  will be saturated with respect to  $(x_0, x_1, \dots, x_r)$  and hence  $R'$  will equal the homogeneous coordinate ring of  $X$ . Choosing  $f_0, f_1, \dots, f_n$  as in Corollary 1.3, it follows that  $B[f_0, f_1, \dots, f_n] \subseteq R'$  is a finite extension, and thus so is  $B[f_0, f_1, \dots, f_n] \subseteq R$ .  $\square$

## 8. EXAMPLES

*Example 8.1.* By Corollary 1.1, it is more difficult to randomly find parameters on surfaces that contain lots of lines. Consider  $\mathbb{V}(xyz) \subset \mathbb{P}^3$  which contains substantially more lines than  $\mathbb{V}(x^2 + y^2 + z^2) \subset \mathbb{P}^3$ . Using Macaulay2 [M2] to select 1,000,000 random pairs  $(f_0, f_1)$  of polynomials of degree two, the proportion that failed to be systems of parameters were:

|                | $\mathbb{V}(xyz)$ | $\mathbb{V}(x^2 + y^2 + z^2)$ |
|----------------|-------------------|-------------------------------|
| $\mathbb{F}_2$ | .2638             | .1179                         |
| $\mathbb{F}_3$ | .0552             | .0059                         |
| $\mathbb{F}_5$ | .0063             | .0004                         |

*Example 8.2.* Let  $X \subseteq \mathbb{P}_{\mathbb{F}_q}^3$  be a smooth cubic surface. Over the algebraic closure  $X$  has 27 lines, but it has between 0 and 27 lines defined over  $\mathbb{F}_q$ . For example, working over  $\mathbb{F}_4$ , the Fermat cubic surface  $X'$  defined by  $x^3 + y^3 + z^3 + w^3$  has 27 lines, while the cubic surface  $X$  defined by  $x^3 + y^3 + z^3 + aw^3$  where  $a \in \mathbb{F}_4 \setminus \mathbb{F}_2$  has no lines defined over  $\mathbb{F}_4$  [DLR15]. It will thus be more difficult to find parameters on  $X$  than on  $X'$ . Using Macaulay2 [M2] to select 100,000 random pairs  $(f_0, f_1)$  of polynomials of degree two, 0.62% failed to be parameters on  $X$  whereas no choices whatsoever failed to be parameters on  $X'$ . This is in line with the predictions from Corollary 1.1; for instance, in the case of  $X$ , we have  $27 \cdot 4^{-2 \cdot 3} \approx 0.66\%$ .

*Example 8.3.* Let  $X = [1 : 4] \cup [3 : 5] \cup [4 : 5] = \mathbb{V}((4x - y)(5x - 3y)(5x - 4y)) \subseteq \mathbb{P}_{\mathbb{Z}}^1$  and let  $R$  be the homogeneous coordinate ring of  $X$ . The fibers are 0-dimensional so finding a Noether normalization  $X \rightarrow \mathbb{P}_{\mathbb{Z}}^0$  is equivalent to finding a single polynomial  $f_0$  that restricts to a unit on all of the points simultaneously. We can find such an  $f_0$  of degree  $d$  if and only if the induced map of free  $\mathbb{Z}$ -modules  $\mathbb{Z}[x, y]_d \rightarrow R_d$  is surjective. A computation in Macaulay2 [M2] shows that this happens if and only if  $d$  is divisible by 60.

*Example 8.4.* Taking the projective closure can fix the failure of finiteness from the affine case. Let  $R = \mathbb{Z}[x]/(3x^2 - 5x) \cong \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{3}]$ . This is a flat, finite type  $\mathbb{Z}$ -algebra where every fiber has dimension 0, yet it is not a finite extension of  $\mathbb{Z}$ . However, if we take the projective closure of  $\text{Spec}(R)$  in  $\mathbb{P}_{\mathbb{Z}}^1$ , then we get  $\text{Proj}(\overline{R})$  where  $\overline{R} = \mathbb{Z}[x, y]/(3x^2 - 5xy)$ . If we then choose  $f_0 := 4x - 7y$ , we see that  $\mathbb{Z}[f_0] \subseteq \overline{R}$  is a finite extension of graded rings.

*Example 8.5.* Let  $\mathbf{k}$  be a field and let  $X = [1 : 1+t] \cup [1-t : 1] = \mathbb{V}((y - (1+t)x)(x - (1-t)y)) \subseteq \mathbb{P}_{\mathbf{k}[t]}^1$ . Let  $R$  be the homogeneous coordinate ring of  $X$ . In degree  $d$ , we have the map  $\phi_d : \mathbf{k}[t][x, y]_d \cong \mathbf{k}[t]^{d+1} \rightarrow R_d \cong \mathbf{k}[t]^2$ . Choosing the standard basis  $x^d, x^{d-1}y, \dots, y^d$  for the source of  $\phi_d$ , and the two points of  $X$  for the target, we can represent  $\phi_d$  by the matrix

$$\begin{pmatrix} 1 & 1+t & (1+t)^2 & \dots & (1+t)^d \\ (1-t)^d & (1-t)^{d-1} & (1-t)^{d-2} & \dots & 1 \end{pmatrix}.$$

It follows that  $\text{im } \phi_d = \text{im } \begin{pmatrix} t^2 & (1+t)^d \\ 0 & 1 \end{pmatrix} = \text{im } \begin{pmatrix} t^2 & 1+dt \\ 0 & 1 \end{pmatrix}$ . The image of  $\phi_d$  thus contains a unit if and only if the characteristic of  $\mathbf{k}$  is  $p$  and  $p|d$ . In particular, if  $\mathbf{k} = \mathbb{Q}$ , then we cannot find a polynomial  $f_0$  inducing a finite map  $X \rightarrow \mathbb{P}_{\mathbb{Q}[t]}^0$ .

*Example 8.6.* Let  $\mathbf{k}$  be any field, let  $B = \mathbf{k}[s, t]$ , and let  $X = [s : 1] \cup [1 : t] = \mathbb{V}((x - sy)(y - tx)) \subseteq \mathbb{P}_B^1$ . We claim that for any  $d > 0$ , there does not exist a polynomial that restricts to a parameter on  $X_b$  for each point  $b \in B$ . Assume for contradiction that we had such an  $f = \sum_{i=0}^d c_i s^i t^{d-i}$  with  $c_i \in B$ . After scaling, we obtain

$$f([s : 1]) = c_0 s^d + c_1 s^{d-1} + \cdots + c_d = 1 \quad \text{and} \quad f([1 : t]) = c_0 + c_1 t + \cdots + c_d t^d = \lambda$$

where  $\lambda \in B^* = \mathbf{k}^*$ . Substituting for  $c_d$  we obtain

$$f([1 : t]) = c_0 + c_1 t + \cdots + c_{d-1} t^{d-1} + (1 - (c_0 s^d + c_1 s^{d-1} + \cdots + c_{d-1} s)) t^d = \lambda,$$

which implies that

$$\begin{aligned} \lambda - t^d &= c_0 + c_1 t + \cdots + c_{d-1} t^{d-1} - (c_0 s^d + c_1 s^{d-1} + \cdots + c_{d-1} s) t^d \\ &= (c_0 - c_0 s^d t^d) + (c_1 t - c_1 s^{d-1} t^d) + \cdots + (c_{d-1} t^{d-1} - c_{d-1} s t^d) = (1 - st)h(s, t) \end{aligned}$$

where  $h(s, t) \in \mathbf{k}[s, t]$ . This implies that  $\lambda - t^d$  is divisible by  $(1 - st)$ , which is a contradiction.

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